J-INTEGRAL CALCULATIONS WITH BOUNDARY ELEMENTS

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INTRODUCTION

In the structural engineering, stress intensity factors are used as an effective criteria to research the fructure mechanics of solid. The precise estimation of stress intensity factors in solving practical problems is an important subject. The boundary element method in this case is expected to be a powerful means of calculation.

In many literatures there appeared the advantage of the boundary integral equation method in comparison with that of domain type methods such as FEM and FDM. Remarkable progresses have been made on boundary type methods under the general title of "Boundary Element Method" by forerunner Brebbia [1].

J-integral, the path independent integral due to Rice [8], is recoginized as an advisable way of calculating stress intensity factors. J-integral is calculated along a path around the crack tip inside the domain. J-integral calculations are expected to have satisfactory results by the boundary element method with solutions of great accuracy inside the domain.

Solutions by direct and indirect boundary element methods were examined in the preliminary study. The indirect method was chosen to compose a J-integral program for two dimensional linear elasticity problems. The present J-integral calculation was applied to some representative problems such as mixed boundary value problems and then the effectiveness of the method was discussed.

PRELIMINARY STUDY, DIRECT AND INDIRECT METHODS

Consider a homogeneous isotropic linear elasticity body occupying a domain V with a boundary S. σ_{ij}, ε_{ij}, T_{i}, U_{i} and summation convention. Superscripts mean;

source ⟶ O □ O ⟵ position

Direct method

Denoting a foundamental solution by superscript *, we have the well known boundary integral equation for boundary value problems;

$$\int {}^*\sigma_{ij,j}\cdot U_i\;dV = \int ({}^*T_i\cdot U_i - {}^*U_i\cdot T_i)\;dS \qquad (1)$$

When the force acting on the foundamental solution is not located inside the domain V, ${}^*\sigma_{ij,j} = 0$ and equation (1) is;

$$\int {}^*T_i\cdot \underline{U_i}\;dS_2 - \int {}^*U_i\cdot \underline{T_i}\;dS_1 = -\int {}^*T_i\cdot U_i\;dS_1 + \int {}^*U_i\cdot T_i\;dS_2 \qquad (2)$$

where S_1 is the essential boundary, S_2 is the natural boundary and the underlines indicate the unknowns. Equation (2) is the equation to calculate the unknown boundary values.

In case of the foundamental solution on which a ℓ-directional unit concentrated force acts at the point K in the domain, ${}^*\sigma_{i\ell,\ell} = -1$ at K and equation (1) is;

$$U^k_\ell = \int ({}^{*k\ell}U_i\cdot T_i - {}^{*k\ell}T_i\cdot U_i)\;dS \qquad (3)$$

Substituting equation (3) into $\sigma_{ij} = C_{ijmn}\frac{1}{2}(U_{m,n}+U_{n,m})$, the stress at the point K is calculated. As an example, expressing σ_x with the plane stress condition, we get;

$$\sigma^k_x = \frac{E}{1-\nu^2}\Big[\int \Big\{\frac{\partial}{\partial X}(-{}^{*kx}U_i) + \nu\frac{\partial}{\partial Y}(-{}^{*ky}U_i)\Big\}\cdot T_i\;dS - \int \Big\{\frac{\partial}{\partial X}(-{}^{*kx}T_i) + \nu\frac{\partial}{\partial Y}(-{}^{*ky}T_i)\Big\}\cdot U_i\;dS\Big] \qquad (4)$$

where E is young's modulus, ν is poisson's ratio and the negatives indicate that the delivatives are not calculated on the boundary but at the point K.

There are two important factors which cause the errors of direct method solutions. The first factor is the difference between simulated boundary values and the theoretical ones of the problem. There are two measures to improve the first factor by adopting the high order elements and by the increase in number of elements. We chose the measure of the increase in number of elements and used simple distributions of the boundary values on the straight segment element. The linear distributed traction and the quadratic distributed displacement indicated in figure 1 were used.

The second factor is the numerical boundary integration.

are able to be utilized for improvement. The boundary integrations have been induced analytically on the elements including the singular point of the foundamental solution due to Brebbia [2]. In the present study, boundary integrations with any conditions of geometry are calculated by analytical formulas due to Hirai [3].

Table 1 is the results of an examination on the singular and regular boundary integration methods. The regular boundary integration method is useful when boundary integrations are calculated numerically. But with the analytical formulas the singular boundary integration method is superior to the regular boundary integration method. Then the singular boundary integration method with the analytical formulas is adopted in the study.

Indirect method

Suppose an exterior domain of an interior domain and opperate the traction $\overline{T}$ which arouses the same displacement as that of the interior domain on the boundary S. Apllying a foundamental solution acting with a ℓ-directional unit force at the point K which is located inside the interior domain, we have two boundary integral equations from equation (1);

$$\text{Interior}\quad U^k_\ell = \int ({}^{*k\ell}U^s_n\cdot T^s_n - {}^{*k\ell}T^s_n\cdot U^s_n)\;dS \qquad (5)$$

$$\text{Exterior}\quad 0 = \int ({}^{*k\ell}U^s_n\cdot \overline{T^s_n} - {}^{*k\ell}\underset{\sim}{T^s_n}\cdot U^s_n)\;dS \qquad (6)$$

Traction; linear　　　Displacement; quadratic

Fig. 1

Table 1

Problems	Without singularity (continuous boundary condition)		With singularity (discontinuous boundary condition)	
Boundary integration	Numerical	Analytical	Numerical	Analytical
Singular	×	○	×	○

where the superscript s indicates the value on the boundary.
$^{*k\ell}\tilde{T}_n^{\,s} = -{}^{*k\ell}T_n^{\,s}$ because the domains are located in oposite sides of the boundary. Then, from equations (5) and (6), we have;

$$U_\ell^k = \int {}^{*k\ell}U_n^s \,(T_n^s + \bar{T}_n^s)\, dS \qquad (7)$$

Substituting equation (7) into $\sigma_{ij} = C_{ijtm}\frac{1}{2}\left(\frac{\partial}{\partial t}U_m + \frac{\partial}{\partial m}U_t\right)$, and $^{*k\ell}U_n^s = {}^{*kn}U_\ell^s$, we have;

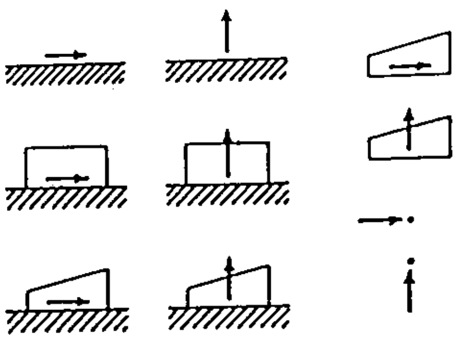
$$\sigma_{ij}^{\;k} = \int \left(-{}^{*kn}\sigma_{ij}^{\;s}\right)\left(T_n^s + \bar{T}_n^s\right) dS \qquad (8)$$

Equations (7) and (8) are the basic formulations of the indirect method.

By the way, $^{*k\ell}U_n^s = {}^{*sn}U_\ell^k$ and $-{}^{*kn}\sigma_{ij}^{\;s} = {}^{*sn}\sigma_{ij}^{\;k}$. Then, replacing $(T_n^s + \bar{T}_n^s)$ by W_n, we get from equations (7) and (8);

$$U_\ell^k = \int {}^{*sn}U_\ell^k \cdot W_n^s \, dS$$
$$\sigma_{ij}^{\;k} = \int {}^{*sn}\sigma_{ij}^{\;k} \cdot W_n^s \, dS \qquad (9)$$

Equations (9) means that the indirect method is numerically equal to the superposition method. The indirect method and the superposition method have been discussed by Hirai [4] [5]. The solutions are calculated by superpositing a certain number of domains in equilibrium of different conditions so as to reproduce the boundary condition. The domains in equilibrium are cut from the foundamental solution. They can be located at any position in the foundamental solution. The unknown W_n means the magnitude of the load acting on the foundamental solutions. The development of the superposition method is easier than that of the indirect boundary integral equation method. Then we treated the indirect method in terms of the principle of superposition.



Semi-infinite Infinite

Fig. 2 shows foundamental solutions used for the indirect method. The stress and displacement of each foundamental solution are calculated analytically. By the superposition of the domains, the boundary conditions were reproduced so as to equate the given conditions on each element concerning with the total and moment.

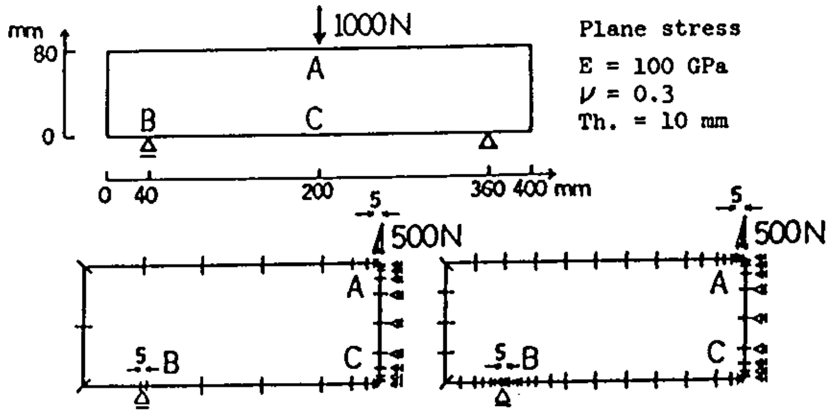
Comparison of solutions

Figure 3 is an example of a beam under center point bending. Figure 4 indicates the solutions of the unknown boundary values at the support B and on the cross section AC. The direct method solutions are the solutions of the simultaneous algebraic equations composed from equation (2). The total amounts of the reacting forces are reasonable but the distributions are insufficient on each element.

The boundary integration of equation (4) is calculated along the whole boundary and the error of the boundary value distribution does not effect considerably on the solution of the stress in the middle position of each element. Then the direct method solutions of stress at the center points of the elements on the cross section AC are relatively sufficient.

A and B in figure 3 have to be treated as singular points. The direct method solutions indicate fatal errors in whole domain when the elements near the singular points are not small. However the indirect method yields satisfactory solutions in whole domain except the little domain near the singular points.

In the present study the indirect method was chosen for the formulation of J-integral.



Indirect Method Direct Method

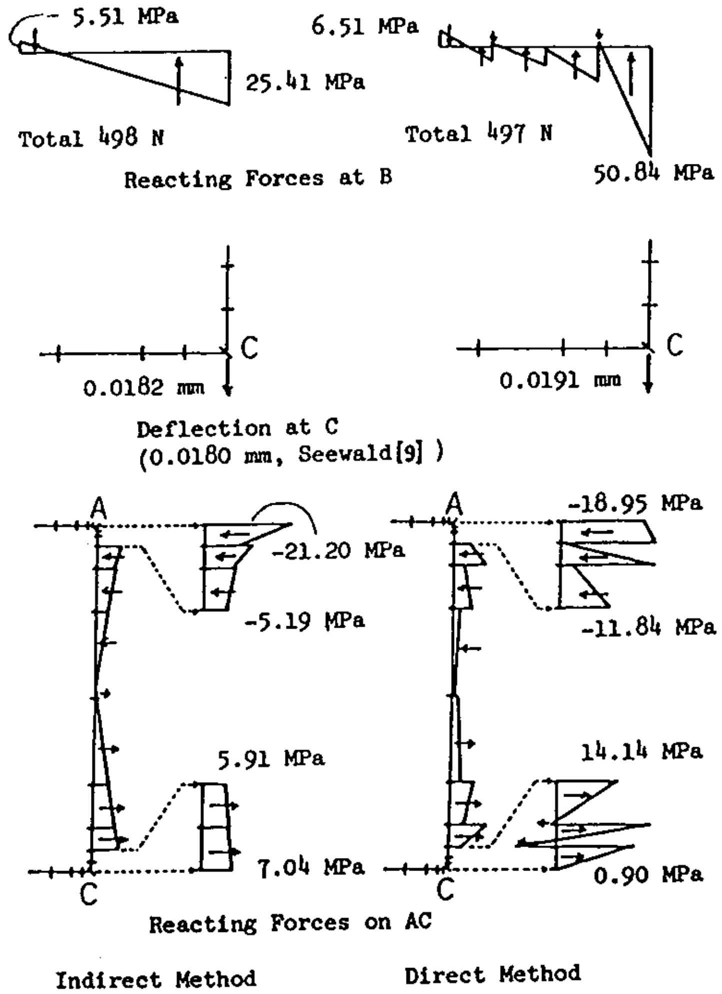


Fig. 4

J-INTEGRAL CALCULATIONS

Formulation with boundary elements

Since J-integral is equal to strain energy release rate for a linear elastic body, stress intensity factors can be related to J-integral by the following equation;

$$J = \frac{\kappa + 1}{8G}\left(K_I^2 + K_{II}^2 \right) + \frac{1}{2G} K_{III}^2 \qquad (10)$$

where J = J-integral

K_I, K_{II}, K_{III} = Stress intensity factors of mode I, II and III

$\kappa = \begin{cases} (3-\nu)/(1+\nu) & \text{plane stress condition} \\ (3-4\nu) & \text{plane strain condition} \end{cases}$

G = Modulus of elasticity in shear

ν = Poisson's ratio

Now, if the coordinate is given as shown in figure 5 and L is an arbitrary counterclockwise path around the crack tip;

$$J = \oint_L \left\{ w\,dy - T_i \frac{\partial U_i}{\partial x} dS \right\} \qquad (11)$$

where w = strain energy density

T = distributed force acting along L from exterior

U = displacement on L

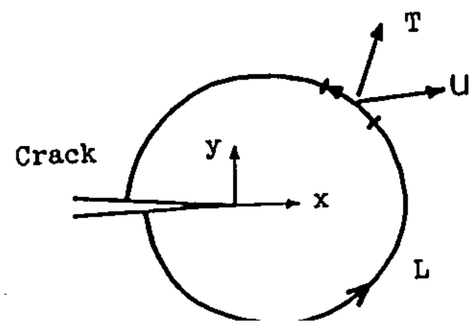
x = axis parallel to the crack surface

When K pieces of domains in equilibrium are superposed to calculate equation (11) by the indirect method, we get;

$$w = \sum^{K} {}^*w \cdot W_n$$
$$T_i = \sum^{K} {}^*T_i \cdot W_n \qquad (12)$$
$$\frac{\partial U_i}{\partial x} = \sum^{K} \frac{\partial {}^*U_i}{\partial x} \cdot W_n$$

Substituting equations (12) to equation (11), we have

$$J = \oint_L \left\{ \left(\sum^{K} {}^*w \cdot W_n\right) dy - \left(\sum^{K} {}^*T_i \cdot W_n\right) \left(\sum^{K} \frac{\partial {}^*U_i}{\partial x} \cdot W_n\right) dS \right\} \qquad (13)$$



Dividing the path L into m strainght line segments to calculate equation (13) numerically, J-integral can be approximately obtained as the sum of discrete values at the center point of each segment, i.e.;

$$J \doteqdot \sum^{m}\left\{\left(\sum^{K} {}^{*}w \cdot W_n\right) \Delta y - \left(\sum^{K} {}^{*}T_i \cdot W_n\right)\left(\sum^{K} \frac{\partial^{*} u_i}{\partial x} \cdot W_n\right) \Delta s\right\} \quad (14)$$

The analytical formulas of the crack directional derivatives of the displacement in the foundamental solutions in figure 2 are derived, and the displacement derivatives are superposed to calculate $\left(\sum^{K} \frac{\partial^{*} u_i}{\partial x} W_n\right)$ in equation (14).

EXAMPLES

In the examinations, the mode I stress intensity factor K_I was calculated with plane stress condition.

A square plate with a central crack under uniform tension

Figure 6 shows the geometry of the problem and the element discretizations of a quater of the domain due to the symmetry.

Stress intensity factors can be calculated by the limiting processes using the stress or displacement near the crack tip in the solution, namely direct stress method and direct displacement method. Figure 7 indicates the results of the J-integral method, the direct stress method and the direct displacement method. The J-integral method is superior to other methods in terms of accuracy.

Table 2 is the results due to the J-integral method with five ratios of crack length to plate width ranging from 0.1 to 0.5. The errors in comparison with the exact solutions are less than 1 %.

A rectangular plate with an edge crack under uniform tension, pure bending and center point bending

The geometries of the problems and the boundary conditions are shown in figure 8. Because of the symmetry, one half of the domain is treated. Table 3 is the J-integral calculations of the problems with several crack length. They agree with the exact solutions within 2 % errors in any cases.

A beam with an edge crack under third point bending subjected to some restraint at the support

The geometry and boundary conditions are shown in figure 9. Four different restraint conditions at the supports are assumed. 100 % restraint is the condition of no horizontal displacement at the support. 66.7 % and 33.3 % restraints are the conditions of 2H / 3 and H / 3 reacting forces at the support. Where H is the horizontal reacting force of 100 % restraint.

Figure 9 is the stress intensity factors calculated by

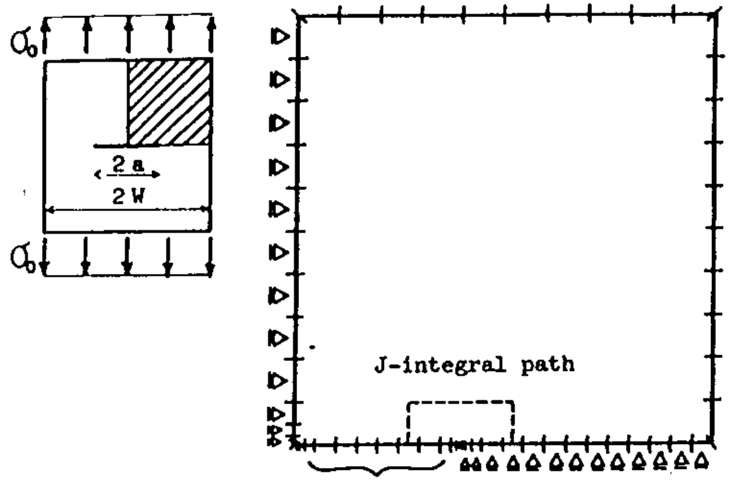


Fig. 6

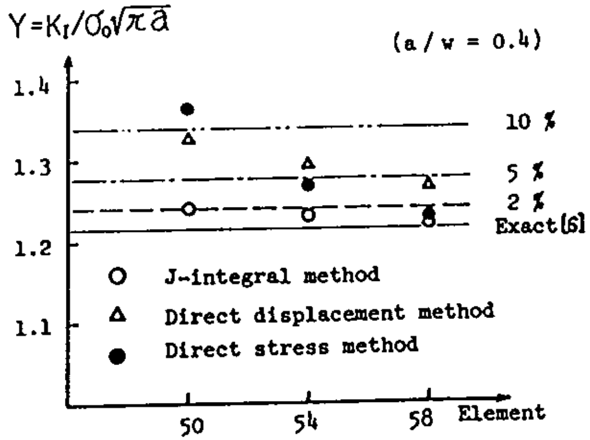


Table 2

$$Y = K_I / \sigma_0 \sqrt{\pi a}$$

a / w	Exact [6]	Present	Error %
0.1	1.014	1.021	+0.69
0.2	1.055	1.061	+0.57
0.3	1.123	1.128	+0.45
0.4	1.216	1.220	+0.33
0.5	1.334	1.335	+0.07

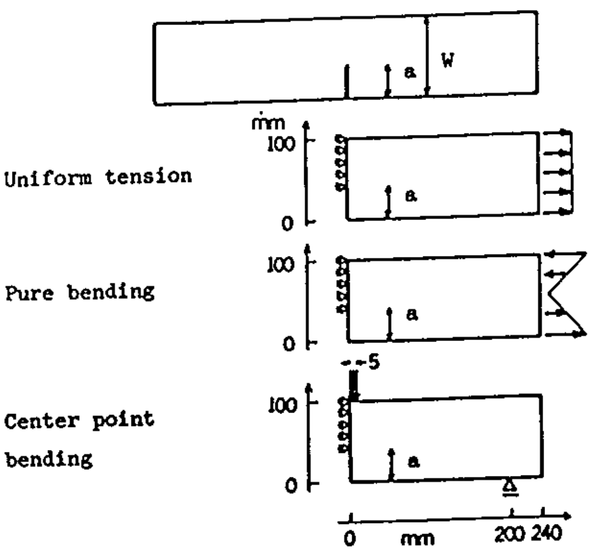


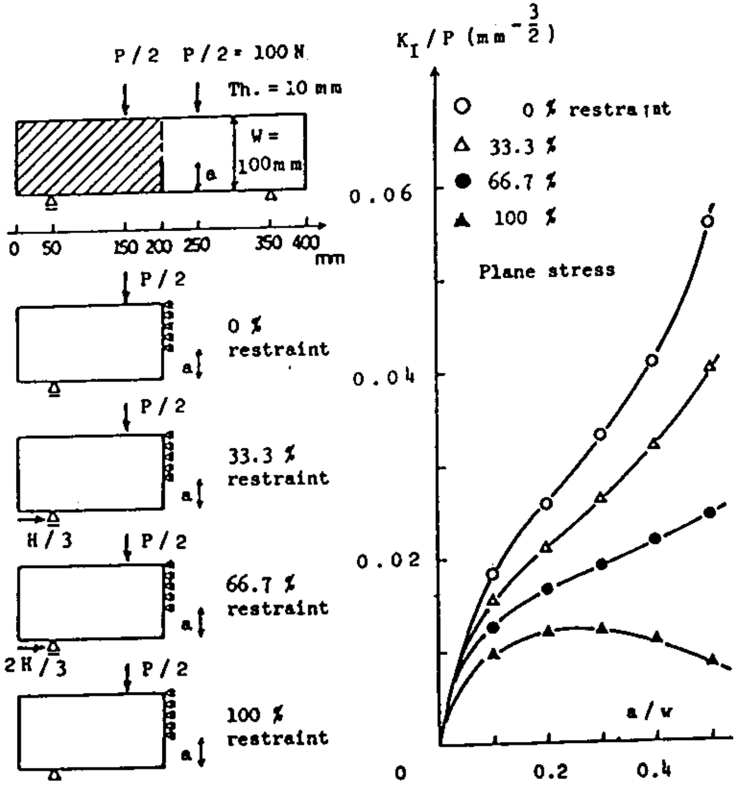
Fig. 8

Table 3

	a / w	Exact [6]	Present	Error %
Uniform tension $Y = K_I / \sigma_t \sqrt{a}$ (σ_t ; tensile stress)	0.1	2.11	2.12	+0.5
	0.2	2.43	2.43	0.0
	0.3	2.95	2.94	-0.3
	0.4	3.74	3.72	-0.5
Pure bending $Y = K_I / \sigma_b \sqrt{a}$ (σ_b ; nominal flexural stress)	0.1	1.852	1.842	-0.54
	0.2	1.869	1.849	-1.07
	0.3	1.992	1.962	-1.51
	0.4	2.229	2.194	-1.60
Center point bending $Y = K_I / \sigma_b \sqrt{a}$ (σ_b ; nominal flexural stress)	0.1	1.746	1.732	-0.81
	0.2	1.738	1.720	-1.05
	0.3	1.848	1.822	-1.43
	0.4	2.080	2.047	-1.61

CONCLUSION

In the preliminary study, the direct and indirect boundary element methods were examined in detail. Satisfactory solutions were calculated by the indirect boundary element method. A J-integral calculation was formulated by the indirect boundary element method and was applied to type I mode stress intensity factors of some representative problems. The present J-integral calculation with boundary elements yielded excellent accuracy and offered an effective measure to estimate stress intensity factors with arbitrary boundary conditions such as mixed boundary value problems.



(H →; 100 % restraint reacting force)

Fig. 9

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